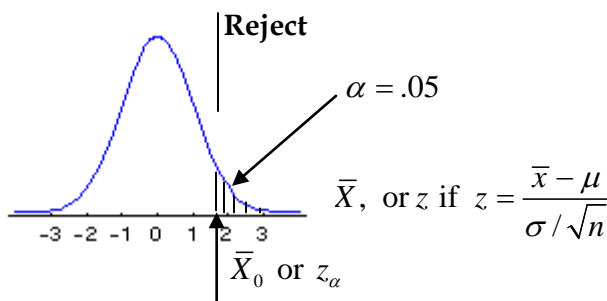


Hypothesis Testing and Interval Estimation. With a test of hypothesis we get the test statistic distribution information from the Null Hypothesis, and then determine the "rejection region" for the test statistic based on the test's significance level α (say 5%). Then if the value we get for our statistic is so extreme that it falls in the rejection region, we say the distributional information (parameters, parametric form, etc.) specified in the null must be rejected.

Relationship to $(1-\alpha)\%$ Confidence Interval. The relationship between a 2-sided significance test and the $(1-\alpha)$ confidence interval is a dual. A level- α significance test rejects $H_0: \mu = \mu_0$ exactly when μ_0 falls outside a $(1-\alpha)$ confidence interval for μ .

Basic Hypothesis Testing and Power of a Test. We provide the basics of hypothesis testing and then calculate the power of a test with regard to a specific alternative hypothesis. We first define our sample data as $X = x_1, x_2, \dots, x_n$. We define a "critical region" W , or rejection region, of "size" (probability), say $\alpha = .05$. We then say $P(X \text{ is in the reject region given null hypothesis}) \leq \alpha = .05$, or $P(X \in W | H_0) \leq \alpha$ which means we reject the null with no more than 5% chance if its true. Let us say α exactly=.05. Our procedure is:

- State the Hypothesis. $H_0: \mu = \mu_0$ (for example)
 $H_a: \mu > \mu_0$ (1-sided test)
- State Size of Test α . This is the significance level, say $\alpha = .05$
- Devise a statistic and its distribution under H_0 : e.g., for $X \sim N(\mu, \sigma^2)$, say $T(x) = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$,
and, assuming σ known, we then have that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, with standard deviation is $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$.
- Draw the Critical Region on the Sampling Distribution specified by the Null:



For $\alpha = .05$, reject if $z_{\alpha} \geq 1.96$ (from the Normal table for $p=.05$), or
 $\bar{X} \geq \bar{X}_0 = \sigma_{\bar{X}} z_{\alpha} + \mu_0$

- Calculate statistic, see if you reject or not. Example: (problem 6.63), SRS $n=500$

$$H_0: \mu = 450$$

$$\alpha = .01 \text{ gives critical value } z_{\alpha} = 2.326.$$

$$H_a: \mu > 450$$

Since $\sigma_{\bar{X}} = 100 / \sqrt{500} = 4.4721$, our critical value in terms of \bar{X} is for $\bar{X} \geq \bar{X}_0 = 4.4721(2.326) + 450 = 460.4$, so we would reject H_0 if our \bar{X} turns out to be greater than 460.4.

f. Power of the Test. The power β of the test is defined as the $P(\text{reject } H_0)$, irrespective of which hypothesis is true; however, in order to calculate these probabilities, we need to know the distribution of the test statistics, which DOES depend on the hypothesis. We require the $P(\text{reject } H_0 \text{ when it is false})$ to be high, i.e. $P(X \in W | H_a)$. We want $\beta(\theta)$ to be as large as possible when H_a is true, and to have the highest power possible for all parameter values in the alternative hypothesis.

In our example, $P(X \in W | H_a) = P(\bar{X} \geq \bar{X}_0 | H_a) = P(\bar{X} \geq 460.4 | H_a)$, so we have to figure this probability in terms of H_a . We can actually calculate the power as a function of all parameter values in the alternative parameter space and plot the power function. In this example we only plot for certain values of the parameter.

Under a specific value for μ_1 in H_a , $z = \frac{\bar{X} - \mu_1}{\sigma_{\bar{X}}}$. Doing some subtraction and division inside the

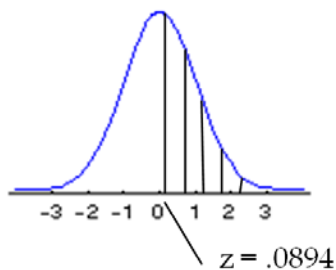
probability statement gives $P(\bar{X} \geq \bar{X}_0 | H_a) = P(\frac{\bar{X} - \mu_1}{\sigma_{\bar{X}}} \geq \frac{\bar{X}_0 - \mu_1}{\sigma_{\bar{X}}})$, but $\frac{\bar{X} - \text{"true" mean}}{\sigma_{\bar{X}}} = z$, so we

obtain $P(z \geq \frac{\bar{X}_0 - \mu_1}{\sigma / \sqrt{n}}) = \beta$, the power of the test for μ_1 , a specific value in the alternative hypothesis.

Example (Problem 6.63, CONT'D): We check for μ_1 in $H_a = 460$.

$$\beta = P(z \geq \frac{\bar{X}_0 - \mu_1}{\sigma_{\bar{X}}}) = P(z \geq \frac{460.4 - 460}{4.4721}) = P(z \geq .0894) = .4644$$

Note: We get this from either the SECOND Z table (1-Positive z values), or by symmetry in the FIRST z table for $z = -.0894$.



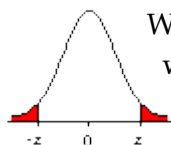
Since we prefer $\beta > .8$ for sensitivity, this test is pretty lame for detecting a 10 point difference. Suppose however that $\mu_1 = 475$, a 25 point difference. Here,

$$\beta_{475} = P(z \geq \frac{460.4 - 475}{4.4721}) = P(z \geq -3.26) = .9994, \text{ practically the whole}$$

normal curve to the right, and we still have $P(\text{reject } H_0 \text{ falsely})$ at only $\alpha = .05$.

For a 2-Sided Test, the same procedure applies. The difference is that the critical region is the union of 2 disjoint sets. For symmetric distributions such as Normal and the "t", we determine $P(X \in W) = \alpha$ as

$$P(X \in W) = P(X \in W_{\text{LEFT}} \text{ or } X \in W_{\text{RIGHT}}) = P(X \in W_{\text{LEFT}}) + P(X \in W_{\text{RIGHT}}) = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$



We then draw our critical region(s) and determine the values of the statistic for which we will reject H_0 . Then power can be calculated as before but with two regions of interest.



Hypothesis Testing and Interval Estimation

Another Complete Example (problem 6.64). Mean amount of Coke in bottle. SRS, $n=1$ sixpack=6 bottles. Assume X is content in bottles is Normal($\mu, \sigma = 3$). Devise a Hypothesis Test on actual contents mean μ .

a. $H_0: \mu = 300$

$H_a: \mu < 300$

b. $\alpha = .05$ gives 1-sided critical value $z_\alpha = -1.645$ (conveniently given)



c. Statistic. \bar{X} under H_0 is $N(\mu_0, \sigma_{\bar{X}}) = N(300, \frac{3}{\sqrt{6}}) = N(300, 1.225)$

d. Critical Region. Reject if $\bar{X} \leq \bar{X}_0 = \sigma_{\bar{X}} z_\alpha + \mu_0 = 1.225(-1.645) + 300 = 297.98$. Repeating, reject if $\bar{X} \leq \bar{X}_0 = 297.98$.

f. Compute Power, ability of the test to reject H_0 if it's false.

μ_1	$\beta = P(z \leq \frac{\bar{X}_0 - \mu_1}{\sigma_{\bar{X}}})$
299	$P(z \leq \frac{297.98 - 299}{1.225}) = P(z \leq -.832) \doteq .202$
295	$P(z \leq \frac{297.98 - 295}{1.225}) = P(z \leq +2.433) \doteq .9926$

Again, for μ_1 close to μ_0 (299 vs. 300), the power is low, but when μ_1 vs. μ_0 is (295 vs. 300), a difference of only 1.6%, the power is almost 1!